Mastermath	Commutative algebra exam	14:00-17:00 14/1/2020

- Write your name, university, and student number on every sheet you hand in.
- You may not use any books or notes during the exam.
- Unless stated otherwise, you need to give full proofs in all your answers. You are allowed to use results that are treated in the book and lectures.
- If you cannot do a part of a question, you may still use its conclusion later on.
- There are **5** questions in total. The exam continues on the back of this sheet.

(1) Let A be a ring and $S \subseteq A$ a subset.

(a) State what it means for $S \subseteq A$ to be *multiplicatively closed*.

Suppose now that S is multiplicatively closed. The natural map $\varphi\colon A \to S^{-1}A$ induces a map $\psi\colon \operatorname{Spec} S^{-1}A \to \operatorname{Spec} A$.

(b) Let $\mathfrak{p}$ be a prime ideal of $S^{-1}A$. Show that the ideal $\mathfrak{q}$ of $S^{-1}A$ generated by $\varphi(\varphi^{-1}(\mathfrak{p}))$ is equal to $\mathfrak{p}$.

It follows immediately that ψ is injective (you do not need to justify this).

(c) Show that the image of ψ consists exactly of those prime ideals of A which have empty intersection with S.

(d) Is the image of ψ always an open subset of $\operatorname{Spec} A$ (with respect to the Zariski topology)? Give a proof or counter-example (with justification).

(2) Let k be a field. At the top of the following table, two rings R, each with an R-algebra A, are listed.

	$R = k[X,Y]$, $A = R[Z]/\langle XZ - Y\rangle$	$R = k[X]$, $A = \frac{R[Y,Z]}{\langle YZ-X\rangle}$
A is a finitely generated R-algebra		
A is a finitely generated R-module		
A is a flat R-module	(b)	(c)

(a) **Copy the above table onto your answer paper**, then fill in each box in the table with T or F, according to whether or not the given property is true for the given ring R, and R-algebra A (sometimes viewed as R-module) in that column. **You do not need to justify your answers to this part.**

(b) Prove your answer in the box marked (b).

(c) Prove your answer in the box marked (c).

There are **5** questions in total. The exam continues on the back of this sheet.

(3) (a) What does it mean for an ideal to be *irreducible*?

(b) (A&M, Thm 7.11) Let A be a noetherian ring, and $I \subseteq A$ an ideal. Show that I is a finite intersection of irreducible ideals.

(c) What does it mean for an ideal to be *primary*?

(d) (A&M, Thm 7.12) Let A be a noetherian ring, and $I \subseteq A$ an irreducible ideal. Show that I is primary.

(4) Let A be a ring and $\mathfrak{a}$ an ideal. Recall that the completion $\hat{A}$ of A at $\mathfrak{a}$ is defined as the inverse limit of the system $A/\mathfrak{a}^n$.

(a) State what it means for A to be *complete* with respect to $\mathfrak{a}$.

(b) Show that every Artin local ring is complete with respect to its maximal ideal.

(c) Let k be a field, $A = k[x, y]$ and $\mathfrak{a} = (x)$. Is the completion $\hat{A}$ a local ring? Justify your answer.

(5) (a) Let A be a ring and $0 \to M' \to M \to M'' \to 0$ an exact sequence of A-modules, with M'' flat. Show that for all A-modules N (including the non-flat ones), the sequence $0 \to M' \otimes N \to M \otimes N \to M'' \otimes N \to 0$ is exact. You may assume without proof that the tensor product is right-exact. *Hint: try writing N as a quotient of a free module (not necessarily of finite rank!).*

(b) (A&M, Ex. 3.15) Let A be a local ring, and let F be the A-module A^n. Show that every set of n generators of F is a basis of F. You may use without proof the following fact: if M is a finitely generated A-module and $\varphi\colon M \to F$ is a surjection, then the kernel of φ is finitely generated.

(c) Deduce that every set of generators of F has at least n elements.